J. Phys. A: Math. Gen. **32** (1999) 8685–8689. Printed in the UK PII: S0305-4470(99)06563-4

Orthonormalized eigenstates of the operator $(\hat{a}f(\hat{n}))^k$ $(k \geqslant 1)$ and their generation

Xiao-Ming Liu

Department of Physics, Beijing Normal University, Beijing 100875, People's Republic of China†
and
CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China

E-mail: Liuxm@263.net

Received 3 August 1999

Abstract. The k orthonormalized eigenstates of the powers $(\hat{a}f(\hat{n}))^k$ $(k \geqslant 1)$ of the annihilation operator $\hat{a}f(\hat{n})$ of f-oscillators are obtained and their properties are discussed. An alternative method to construct them is proposed, and the result shows that all of the eigenstates can be generated by a linear superposition of k f-coherent states.

1. Introduction

Recently, there has been much interest in the study of nonlinear coherent states called f-coherent states [1], which are eigenstates of the annihilation operator $\hat{a}f(\hat{n})$ of f-oscillators. A class of f-coherent states can be realized physically as the stationary states of the centre-of-mass motion of a trapped ion [2]. The f-coherent states exhibit non-classical features such as squeezing and self-splitting. Subsequently, even and odd f-coherent states, which are orthonormalized eigenstates of the square $(\hat{a}f(\hat{n}))^2$ of the operator $\hat{a}f(\hat{n})$, were constructed and their non-classical effects were studied [3, 4]. In this paper, we will construct orthonormalized eigenstates of the high powers $(\hat{a}f(\hat{n}))^k$ $(k \geqslant 1)$ of the operator $\hat{a}f(\hat{n})$, discuss their properties and explore their generation in terms of f-coherent states.

2. The k orthonormalized eigenstates of $(\hat{a}f(\hat{n}))^k$

The annihilation operator A and the creation operator A^+ of f-oscillators are distortions of the annihilation and creation operators $\hat{a}$ and $\hat{a}^+$ of the usual harmonic oscillator, and are given by [1, 2]

$$A = \hat{a}f(\hat{n}) = f(\hat{n}+1)\hat{a} \tag{1}$$

$$A^+ = f^+(\hat{n})\,\hat{a}^+ = \hat{a}^+ f^+(\hat{n}+1) \tag{2}$$

where

$$\hat{n} = \hat{a}^+\hat{a} \qquad [\hat{n}, A] = -A \qquad [\hat{n}, A^+] = A^+ \tag{3}$$

where f is an operator-valued function of the number operator $\hat{n}$.

† Mailing address.

The commutator between A and A^+ can be easily computed by the relations

$$A = \sum_{n=0}^{\infty} \sqrt{n} f(n) |n-1\rangle\langle n| \tag{4}$$

$$A^+ = \sum_{n=0}^{\infty} \sqrt{n} f^*(n) |n\rangle\langle n-1| \tag{5}$$

and it reads

$$[A, A^+] = (\hat{n}+1) f^2(\hat{n}+1) - \hat{n} f^2(\hat{n}) \tag{6}$$

where f is chosen to be real and $f^+(\hat{n}) = f(\hat{n})$.

Let us consider the following states:

$$|\psi_j(\alpha, f)\rangle_k = C_j \sum_{n=0}^{\infty} \frac{\alpha^{kn+j}}{\sqrt{(kn+j)!} f(kn+j)!} |kn+j\rangle \tag{7}$$

with

$$f(kn+j)! = f(kn+j) f(kn+j-1) \ldots f(1) f(0) \tag{8}$$

where k is a positive integer ($k = 1, 2, 3, \ldots$); $j = 0, 1, \ldots, k-1$; C_j are normalized factors and α is a complex parameter. With A^k operating on $|\psi_j(\alpha, f)\rangle_k$, we have

$$\begin{aligned} A^k |\psi_j(\alpha, f)\rangle_k &= \alpha^k C_j \sum_{n=0}^{\infty} \frac{\alpha^{kn+j}}{\sqrt{(kn+j)!} f(kn+j)!} |kn+j\rangle \\ &= \alpha^k |\psi_j(\alpha, f)\rangle_k. \end{aligned} \tag{9}$$

As a result, the k states of (7) are all the eigenstates of the operator A^k with the same eigenvalue α^k. It is easy to check that, for the same value of k, these states are orthogonal to each other with respect to the subscript j

$${}_k\langle\psi_i(\alpha, f)|\psi_j(\alpha', f)\rangle_k = 0 \qquad (i, j = 0, 1, \ldots, k-1, i \neq j). \tag{10}$$

Let $|\alpha|^2 = x$. We easily suppose C_j to be real number. Using the normalized conditions

$${}_k\langle\psi_j(\alpha, f)|\psi_j(\alpha, f)\rangle_k = C_j^2 \sum_{n=0}^{\infty} \frac{x^{kn+j}}{(kn+j)! [f(kn+j)!]^2} = C_j^2 A_j(x, f) = 1. \tag{11}$$

We have

$$C_j = A_j^{-1/2}(x, f) \tag{12}$$

where

$$A_j(x, f) = \sum_{n=o}^{\infty} \frac{x^{kn+j}}{(kn+j)! [f(kn+j)!]^2}. \tag{13}$$

From (13) it follows that

$$\sum_{j=0}^{k-1} A_j(x, f) = \sum_{n=0}^{\infty} \frac{x^n}{n! [f(n)!]^2} \equiv e_f(x). \tag{14}$$

It should be noted that the k states $|\psi_j(\alpha, f)\rangle_k$ are normalizable provided C_j are non-zero and finite. This means that the terms in summation for $A_j(x, f)$ should be such that

$$|\alpha|^2 < \lim_{n\to\infty} n f^2(n). \tag{15}$$

If $f(n)$ decreases faster than $n^{-1/2}$ for large n, then the range of α, for which the $|\psi_j(\alpha, f)\rangle_k$ are normalizable, is restricted to values satisfying (15) and in other cases the range of α is unrestricted.

We may obtain

$$A|\psi_j(\alpha, f)\rangle_k = \alpha A_j^{-1/2}(|\alpha|^2, f) A_{j-1}^{1/2}(|\alpha|^2, f)|\psi_{j-1}(\alpha, f)\rangle_k \qquad j = 1, 2, \ldots, k-1 \tag{16}$$

$$A^i|\psi_0(\alpha, f)\rangle_k = \alpha^i A_0^{-1/2}(|\alpha|^2, f) A_{k-i}^{1/2}(|\alpha|^2, f)|\psi_{k-i}(\alpha, f)\rangle_k \qquad i = 1, 2, \ldots, k. \tag{17}$$

It indicates that, by the successive actions of the operator A, the k eigenstate vectors of A^k can be transformed into each other in this way: $|\psi_0\rangle_k \to |\psi_{k-1}\rangle_k \to |\psi_{k-2}\rangle_k \to \cdots \to |\psi_1\rangle_k \to |\psi_0\rangle_k$. Actually, the operator A plays the role of a rotating operator in the k eigenstate vectors of A^k.

The definition of f-coherent states [1] is

$$|\alpha, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}f(n)!}|n\rangle \tag{18}$$

with

$$N_f = (e_f(|\alpha|^2))^{-1/2}. \tag{19}$$

In terms of the k eigenstates $|\psi_j(\alpha, f)\rangle_k$ of A^k, the f-coherent states can be expanded in this way

$$|\alpha, f\rangle = N_f \left[\sum_{j=0}^{k-1} A_j^{1/2}(|\alpha|^2, f)|\psi_j(\alpha, f)\rangle_k\right]. \tag{20}$$

Note that $|\alpha, f\rangle$ and $|\psi_j(\alpha, f)\rangle_k$ are non-trivially different.

We should emphasize that here we discuss orthogonality of $|\psi_j(\alpha, f)\rangle_k$ with respect to the subscript j. For $\alpha \neq \alpha'$, we obtain

$$\begin{aligned} {}_k\langle\psi_j(\alpha, f)|\psi_j(\alpha', f)\rangle_k &= \left[A_j(|\alpha|^2, f)A_j(|\alpha'|^2, f)\right]^{-1/2} \sum_{n=0}^{\infty} \frac{(\alpha^*\alpha')^{kn+j}}{(kn+j)![f(kn+j)!]^2} \\ &= \left[A_j(|\alpha|^2, f)A_j(|\alpha'|^2, f)\right]^{-1/2} A_j(\alpha^*\alpha', f) \neq 0. \end{aligned} \tag{21}$$

Therefore, when $\alpha \neq \alpha'$, $|\psi_j(\alpha, f)\rangle_k$ and $|\psi_j(\alpha', f)\rangle_k$ are not orthogonal.

As $k = 1$, $|\psi_0(\alpha, f)\rangle_1$ are exactly the f-coherent states.

As two special cases, for $f(\hat{n}) \to \hat{1}$, $|\psi_j(\alpha, f)\rangle_k$ become k orthonormalized eigenstates of the high powers of the annihilation operator of the usual harmonic oscillator [5]; for $f(\hat{n}) \to \sqrt{(q^{\hat{n}} - q^{-\hat{n}})/(q - q^{-1})\hat{n}}$ (where q is a continuous parameter), $|\psi_j(\alpha, f)\rangle_k$ become k orthonormalized eigenstates of that of the q-deformed harmonic oscillator [6].

It is interesting to note that Klauder and co-workers have studied an extremely wide class of coherent states that includes the f-coherent states as a small subset [7–9]. However, the k orthonormalized eigenstates of A^k are different from the Klauder-type coherent states. The k states can also be obtained by considering a suitable linear superposition of the Klauder-type states.

3. Generation of the k orthonormalized eigenstates of $(\hat{a}f(\hat{n}))^k$

According to (20), we consider the following k f-coherent states:

$$|\alpha_l, f\rangle = |\alpha e^{i2\pi l/k}, f\rangle$$
$$= e_f^{-1/2}(|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}f(n)!}e^{i(2\pi/k)ln}|n\rangle \qquad l = 0, 1, \ldots, k-1. \tag{22}$$

The k f-coherent states are discretely distributed with an equal interval of angle along a circle around the origin of the α-plane. The inner product of the two states of (22) is

$$\langle\alpha_l, f|\alpha_{l'}, f\rangle = e_f^{-1}(|\alpha|^2)e_f\left(|\alpha|^2 e^{i2\pi(l'-l)/k}\right) \qquad (l, l' = 0, 1, \ldots, k-1). \tag{23}$$

Consider a linear transformation S such that

$$|\varphi\rangle_k = S|\alpha, f\rangle_k \tag{24}$$

where

$$|\alpha, f\rangle_k = \begin{bmatrix} |\alpha_0, f\rangle \\ |\alpha_1, f\rangle \\ \vdots \\ |\alpha_{k-1}, f\rangle \end{bmatrix} \qquad |\varphi\rangle_k = \begin{bmatrix} |\varphi_0\rangle_k \\ |\varphi_1\rangle_k \\ \vdots \\ |\varphi_{k-1}\rangle_k \end{bmatrix}. \tag{25}$$

S is a $k\times k$ matrix that makes φ_j orthonormal, and ${}_k\langle\varphi_j|\varphi_{j'}\rangle_k = \delta_{jj'}$. The above requirement leads to a set of algebraic equations for S_{ij},

$$\sum_{l=0}^{k-1}\sum_{l'=0}^{k-1} e_f^{-1}\left(|\alpha|^2\right)e_f\left(|\alpha|^2 e^{i(2\pi/k)(l'-l)}\right) S_{jl}^* S_{j'l'} = \delta_{jj'}. \tag{26}$$

The solution of equation (26), S_{ij}, can be found as follows. By virtue of the relation

$$\sum_{l'=0}^{k-1} e_f\left(|\alpha|^2 e^{\pm i(2\pi/k)(l'-l)}\right) e^{-i(2\pi/k)jl'} = e^{-i(2\pi/k)jl}\sum_{l'=0}^{k-1} e_f\left(|\alpha|^2 e^{\pm i(2\pi/k)l'}\right) e^{-i(2\pi/k)jl'} \tag{27}$$

the matrix elements of S that satisfy (26) are given by

$$S_{jl} = \frac{1}{k}e_f^{1/2}(|\alpha|^2)\left[\frac{1}{k}\sum_{l'=0}^{k-1} e_f\left(|\alpha|^2 e^{i(2\pi/k)l'}\right) e^{-i(2\pi/k)jl'}\right]^{-1/2} e^{-i(2\pi/k)jl}$$
$$= \frac{1}{k}e_f^{1/2}(|\alpha|^2)A_j^{-1/2}(|\alpha|^2, f)\, e^{-i(2\pi/k)jl} \qquad (j, l = 0, 1 \ldots, k-1). \tag{28}$$

From (24) and (28), we obtain k orthonormalized states

$$|\varphi_j\rangle_k = \frac{1}{k}A_j^{-1/2}(|\alpha|^2, f)e_f^{1/2}(|\alpha|^2)\sum_{l=0}^{k-1} e^{-i(2\pi/k)jl}\left|\alpha e^{i(2\pi/k)l}, f\right\rangle \qquad j = 0, 1, \ldots, k-1 \tag{29}$$

which are just what we want. By use of the relation

$$\sum_{l=0}^{k-1} e^{i(2\pi/k)lt} = 0 \qquad t = 1, 2, \ldots, k-1 \tag{30}$$

it can be proved that

$$|\varphi_j\rangle_k = |\psi_j(\alpha, f)\rangle_k \qquad j = 0, 1, \ldots, k-1. \tag{31}$$

According to (29), for $k = 2$, we obtain

$$|\varphi_0\rangle_2 = \tfrac{1}{2}A_0^{-1/2}(|\alpha|^2, f)\, e_f^{1/2}(|\alpha|^2)(|\alpha, f\rangle + |-\alpha, f\rangle) \tag{32}$$

$$|\varphi_1\rangle_2 = \tfrac{1}{2}A_1^{-1/2}(|\alpha|^2, f)\, e_f^{1/2}(|\alpha|^2)(|\alpha, f\rangle - |-\alpha, f\rangle) \tag{33}$$

which are just the so-called even and odd f-coherent states studied in [3].

The $|\varphi_j\rangle_k$ $(j = 0, 1, \ldots, k-1)$ in (29) are exactly the k orthonormalized eigenstates of $(\hat{a}f(\hat{n}))^k$ obtained in section 2, but reconstructed here by a different method. From the above reconstruction, we come to an important conclusion that any orthonormalized eigenstates of $(\hat{a}f(\hat{n}))^k$ can be generated from a linear superposition of k f-coherent states $|\alpha e^{i(2\pi/k)l}, f\rangle$ $(l = 0, 1, \ldots, k-1)$, which have the same amplitude but different phases. Yet, from (29), one can find the connection between f-coherent states and these k eigenstates.

4. Summary

We have derived the k orthonormalized eigenstates of the powers $(\hat{a}f(\hat{n}))^k$ $(k \geqslant 1)$ of the annihilation operator $\hat{a}f(\hat{n})$ of f-oscillators, and discussed their properties. An alternative method to construct such eigenstates is proposed, and we come to an important conclusion that all of them can be generated by a linear superposition of k f-coherent states that have the same amplitude but different phases.

Acknowledgment

I would like to thank the referees for suggestions leading to the improvement of the paper.

References

[1] Manko V I, Marmo G, Zaccaria F and Sudarshan E C G 1997 *Phys. Scr.* **55** 528
[2] de Matos Filho R L and Vogel W 1996 *Phys. Rev.* A **54** 4560
[3] Mancini S 1997 *Phys. Lett.* A **233** 291
[4] Sivakumar S 1989 *Phys. Lett.* A **250** 257
[5] Sun J, Wang J and Wang C 1991 *Phys. Rev.* A **44** 3369
[6] Liu X-M, Wang S-J and Zhang W-Z 1995 *Phys. Rev.* A **51** 4929
Liu X-M 1994 *PhD Thesis* Lanzhou University, China
[7] Klauder J R 1993 *Mod. Phys. Lett.* A **8** 1735
[8] Klauder J R 1995 *Ann. Phys.* **237** 147
[9] Gazeau J P and Klauder J R 1999 *J. Phys. A: Math. Gen.* **32** 123